

On the Stanley depth of edge ideals of line and cyclic graphs

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Abstract

We prove that the edge ideals of line and cyclic graphs and their quotient rings satisfy the Stanley conjecture. We compute the Stanley depth for the quotient ring of the edge ideal associated to a cycle graph of length n , given a precise formula for $n \equiv 0, 2 \pmod{3}$ and tight bounds for $n \equiv 1 \pmod{3}$. Also, we give bounds for the Stanley depth of a quotient of two monomial ideals, in combinatorial terms.

Keywords: Stanley depth, Stanley conjecture, monomial ideal, edge ideal.

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Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K . Let M be a \mathbb{Z}^n -graded S -module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as a \mathbb{Z}^n -graded K -vector space, where $m_i \in M$ is homogeneous with respect to \mathbb{Z}^n -grading, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$ -submodule of M . We define $\text{sdepth}(\mathcal{D}) = \min_{i=1, \dots, r} |Z_i|$ and $\text{sdepth}_S(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\text{sdepth}_S(M)$ is called the *Stanley depth* of M . In [1], J. Apel restated a conjecture firstly given by Stanley in [15], namely that $\text{sdepth}_S(M) \geq \text{depth}_S(M)$ for any \mathbb{Z}^n -graded S -module M . This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals, see [9].

Herzog, Vladioiu and Zheng show in [10] that $\text{sdepth}_S(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. However, it is difficult to compute this invariant, even in some very particular cases. In [14], Rinaldo give a computer implementation for this algorithm, in the computer algebra system *CoCoA* [8]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [2] Biro et al. proved that $\text{sdepth}(m) = \lceil n/2 \rceil$ where $m = (x_1, \dots, x_n)$.

Let I_n and J_n be the edges ideals associated to the n -line, respectively n -cycle, graph. Firstly, we prove that $\text{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$, see Proposition 1.3. Alin Ştefan [16] proved that $\text{sdepth}(S/I_n) = \lceil \frac{n}{3} \rceil$. Using similar techniques, we prove that $\text{sdepth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$, for $n \equiv 0 \pmod{3}$ and $n \equiv 2 \pmod{3}$. Also, we prove that $\lceil \frac{n-1}{3} \rceil \leq \text{sdepth}(S/J_n) \leq \lceil \frac{n}{3} \rceil$, for $n \equiv 1 \pmod{3}$. See Theorem 1.9. In particular, S/J_n satisfies the Stanley conjecture. Also, we note that both I_n and J_n satisfy the Stanley conjecture, see Corollary 1.5. In Proposition 1.10, we prove that $\text{sdepth}(J_n/I_n) = \text{depth}(J_n/I_n) = \lceil \frac{n+2}{3} \rceil$. In the second section, we give an upper bound for the Stanley depth of a quotient of two square free monomial ideals, in combinatorial terms, see Theorem 2.4. Also, we give a lower bound for the Stanley depth of a quotient of two arbitrary monomial ideals, see Proposition 2.9.

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1 Main results

Let $n \geq 3$ be an integer and let $G = (V, E)$ be a graph with the vertex set $V = [n]$ and edge set E . Then the *edge ideal* $I(G)$ associated to G is the squarefree monomial ideal $I = (x_i x_j : \{i, j\} \in E)$ of S .

We consider the *line graph* L_n on the vertex set $[n]$ and with the edge set $E(L_n) = \{(i, i+1) : i \in [n-1]\}$. Then $I_n = I(L_n) = (x_1 x_2, \dots, x_{n-1} x_n) \subset S$. Also, we consider the cyclic graph C_n on the vertex set $[n]$ and with the edge set $E(C_n) = \{(i, i+1) : i \in [n-1]\} \cup \{(n, 1)\}$. Then $J_n = I_n + (x_n x_1) \subset S$.

We recall the well known Depth Lemma, see for instance [18, Lemma 1.3.9] or [17, Lemma 3.1.4].

Lemma 1.1. (*Depth Lemma*) *If $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of modules over a local ring S , or a Noetherian graded ring with S_0 local, then*

- a) $\text{depth } M \geq \min\{\text{depth } N, \text{depth } U\}$.
- b) $\text{depth } U \geq \min\{\text{depth } M, \text{depth } N + 1\}$.
- c) $\text{depth } N \geq \min\{\text{depth } U - 1, \text{depth } M\}$.

Using Depth Lemma, Morey proved in [11] the following result.

Lemma 1.2. [11, Lemma 2.8] $\text{depth}(S/I_n) = \lceil \frac{n}{3} \rceil$.

In the following, we will prove a similar result for S/J_n .

Proposition 1.3. $\text{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$.

Proof. We denote $S_k := K[x_1, \dots, x_k]$, the ring of polynomials in k variables. We use induction on n . If $n \leq 3$ then is an easy exercise to prove the formula. Assume $n \geq 4$ and consider the short exact sequence

$$0 \longrightarrow S/(J_n : x_n) \xrightarrow{\cdot x_n} S/J_n \longrightarrow S/(J_n, x_n) \longrightarrow 0.$$

Note that $(J_n : x_n) = (x_1, x_{n-1}, x_2 x_3, \dots, x_{n-3} x_{n-2})$ and therefore we get $S/(J_n : x_n) \cong K[x_2, \dots, x_{n-2}, x_n]/(x_2 x_3, \dots, x_{n-3} x_{n-2}) \cong (S_{n-3}/I_{n-3})[x_n]$.

Also, $(J_n, x_n) = (x_1 x_2, \dots, x_{n-2} x_{n-1}, x_n)$ and therefore $S/(J_n, x_n) \cong S_{n-1}/I_{n-1}$. By Lemma 1.2, we get $\text{depth}(S/(J_n : x_n)) = \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$ and $\text{depth}(S/(J_n, x_n)) = \lceil \frac{n-1}{3} \rceil$. If $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$, then $\lceil \frac{n-1}{3} \rceil = \lceil \frac{n}{3} \rceil$, and, by using Lemma 1.1, we get $\text{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$, as required.

Assume $n \equiv 1 \pmod{3}$. We claim that we have the S -module isomorphism

$$\frac{(J_n : x_n)}{J_n} \cong x_{n-1} \left(\frac{K[x_1, \dots, x_{n-3}]}{(x_1 x_2, \dots, x_{n-4} x_{n-3})} \right) [x_{n-1}] \oplus x_1 \left(\frac{K[x_3, \dots, x_{n-2}]}{(x_3 x_4, \dots, x_{n-3} x_{n-2})} \right) [x_1].$$

Indeed, if $u \in (J_n : x_n)$ is a monomial such that $u \notin J_n$, then $x_1 | u$ or $x_{n-1} | u$. If $x_{n-1} | u$, then $u = x_{n-1} v$ with $v \in S$.

Since $u \notin J_n$, it follows that $v = x_{n-1}^\alpha w$, with $\alpha \geq 1$, $w \in K[x_1, \dots, x_{n-3}]$ and $w \notin (x_1x_2, \dots, x_{n-4}x_{n-3})$. Similarly, if $x_{n-1} \nmid u$, then $x_1 \mid u$ and $u = x_1^\alpha w$ with $\alpha \geq 1$, $w \in K[x_3, \dots, x_{n-2}]$ and $w \notin (x_3x_4, \dots, x_{n-3}x_{n-2})$.

Using the above isomorphism and Lemma 1.2, it follows that

$$\text{depth} \left(\frac{(J_n : x_n)}{J_n} \right) = \text{depth} \left(\frac{K[x_3, \dots, x_{n-2}]}{(x_3x_4, \dots, x_{n-3}x_{n-2})} \right) + 1 = \left\lceil \frac{n-4}{3} \right\rceil + 1 = \left\lceil \frac{n-1}{3} \right\rceil.$$

Now, using Lemma 1.1 for the short exact sequence $0 \rightarrow \frac{(J_n : x_n)}{J_n} \rightarrow S/J_n \rightarrow S/(J_n : x_n) \rightarrow 0$, we are done. \square

Note that the previous Proposition can be seen as a consequence of [3, Proposition 5.0.6]. However, we preferred to give a direct proof in order to relate it with the Stanley depth case. Now, we recall the following result of Okazaki.

Theorem 1.4. [12, Theorem 2.1] *Let $I \subset S$ be a monomial ideal (minimally) generated by m monomials. Then:*

$$\text{sdepth}(I) \geq \max\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\}.$$

As a direct consequence of Lemma 1.2, Proposition 1.3 and Theorem 1.4, we get.

Corollary 1.5. $\text{sdepth}(I_n) \geq 1 + \frac{n-1}{2}$ and $\text{sdepth}(J_n) \geq \frac{n}{2}$. In particular, I_n and J_n satisfy the Stanley conjecture.

In [16], Alin Ştefan computed the Stanley depth for S/I_n .

Lemma 1.6. [16, Lemma 4] $\text{sdepth}(S/I_n) = \left\lceil \frac{n}{3} \right\rceil$.

In [13], Asia Rauf proved the analog of Lemma 1.1(a) for sdepth :

Lemma 1.7. *Let $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of \mathbb{Z}^n -graded S -modules. Then:*

$$\text{sdepth}(M) \geq \min\{\text{sdepth}(U), \text{sdepth}(N)\}.$$

Using these lemmas, we are able to prove the following Proposition.

Proposition 1.8. $\text{sdepth}(S/J_n) \geq \left\lceil \frac{n-1}{3} \right\rceil$. In particular, S/J_n satisfies the Stanley conjecture.

Proof. As in the proof of Proposition 1.3, we consider the short exact sequence

$$0 \longrightarrow S/(J_n : x_n) \xrightarrow{x_n} S/J_n \longrightarrow S/(J_n, x_n) \longrightarrow 0.$$

Since $S/(J_n : x_n) \cong (S_{n-2}/I_{n-2})[x_n]$ and $S/(J_n, x_n) \cong S_{n-1}/I_{n-1}$, by Lemma 1.6 and [10, Lemma 3.6], we get $\text{sdepth}(S/(J_n : x_n)) = \left\lceil \frac{n-3}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil$ and $\text{sdepth}(S/(J_n, x_n)) = \left\lceil \frac{n-1}{3} \right\rceil$. Using Lemma 1.7, we get $\text{sdepth}(S/J_n) \geq \left\lceil \frac{n-1}{3} \right\rceil$, as required. \square

Let $\mathcal{P} \subset 2^{[n]}$ be a poset and $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of \mathbf{P} . We denote $\text{sdepth}(\mathbf{P}) := \min_{i \in [r]} |D_i|$. Also, we define the Stanley depth of \mathcal{P} , to be the number

$$\text{sdepth}(\mathcal{P}) = \max\{\text{sdepth}(\mathbf{P}) : \mathbf{P} \text{ is a partition of } \mathcal{P}\}.$$

We recall the method of Herzog, Vladoiu and Zheng [10] for computing the Stanley depth of S/I and I , where I is a squarefree monomial ideal. Let $G(I) = \{u_1, \dots, u_s\}$ be the set of minimal monomial generators of I . We define the following two posets:

$$\mathcal{P}_I := \{\sigma \subset [n] : u_i | x_\sigma := \prod_{j \in \sigma} x_j \text{ for some } i\} \text{ and } \mathcal{P}_{S/I} := 2^{[n]} \setminus \mathcal{P}_I.$$

Herzog Vladoiu and Zheng proved in [10] that $\text{sdepth}(I) = \text{sdepth}(\mathcal{P}_I)$ and $\text{sdepth}(S/I) = \text{sdepth}(\mathcal{P}_{S/I})$. Now, for $d \in \mathbb{N}$ and $\sigma \in \mathcal{P}$, we denote

$$\mathcal{P}_d = \{\tau \in \mathcal{P} : |\tau| = d\}, \quad \mathcal{P}_{d,\sigma} = \{\tau \in \mathcal{P}_d : \sigma \subset \tau\}.$$

With these notations, we are able to prove the following result.

Theorem 1.9. (1) $\text{sdepth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$, for $n \equiv 0 \pmod{3}$ and $n \equiv 2 \pmod{3}$.
(2) $\text{sdepth}(S/J_n) \leq \lceil \frac{n}{3} \rceil$, for $n \equiv 1 \pmod{3}$.

Proof. Using Proposition 1.8, it is enough to prove the " \leq " inequalities. Let $\mathcal{P} = \mathcal{P}_{S/J_n}$. Firstly, note that if $\sigma \in \mathcal{P}$ such that $\mathcal{P}_{d,\sigma} = \emptyset$, then $\text{sdepth}(\mathcal{P}) < d$. Indeed, let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of \mathcal{P} with $\text{sdepth}(\mathcal{P}) = \text{sdepth}(\mathbf{P})$. Since $\sigma \in \mathcal{P}$, it follows that $\sigma \in [F_i, G_i]$ for some i . If $|G_i| \geq d$, then it follows that $\mathcal{P}_{\sigma,d} \neq \emptyset$, since there are subsets in the interval $[F_i, G_i]$ of cardinality d which contain σ , a contradiction. Thus, $|G_i| < d$ and therefore $\text{sdepth}(\mathcal{P}) < d$.

We have three cases to study.

1. If $n = 3k \geq 3$ and $\sigma = \{1, 4, \dots, 3k-2\}$, then $\mathcal{P}_{k+1,\sigma} = \emptyset$. Indeed, if $u = x_1 x_4 \cdots x_{3k-2}$, one can easily see that $u \cdot x_j \in J_n$ for all $j \in [n] \setminus \sigma$. Therefore, by previous remark, $\text{sdepth}(S/J_n) = \text{sdepth}(\mathcal{P}) \leq k = \lceil \frac{n-1}{3} \rceil$, as required.
2. If $n = 3k+2 \geq 5$ and $\sigma = \{1, 4, \dots, 3k+1\}$, then $\mathcal{P}_{k+2,\sigma} = \emptyset$. As above, it follows that $\text{sdepth}(S/J_n) \leq k+1 = \lceil \frac{n-1}{3} \rceil$.
3. If $n = 3k+1 \geq 7$ and $\sigma = \{1, 4, \dots, 3k-2, 3k\}$, then $\mathcal{P}_{k+2,\sigma} = \emptyset$ and therefore $\text{sdepth}(\mathcal{P}) \leq k+1 = \lceil \frac{n}{3} \rceil$. \square

Proposition 1.10. $\text{sdepth}(J_n/I_n) = \text{depth}(J_n/I_n) = \lceil \frac{n+2}{3} \rceil$, for all $n \geq 3$.

Proof. One can easily check that $\frac{J_3}{I_3} \cong x_1 x_3 K[x_1, x_3]$. Thus $\text{sdepth}(J_3/I_3) = \text{depth}(J_3/I_3) = 2$, as required. Similarly, for $n = 4$, we have $\frac{J_4}{I_4} \cong x_1 x_4 K[x_1, x_4]$ and for $n = 5$, we have $\frac{J_5}{I_5} \cong x_1 x_5 K[x_1, x_3, x_5]$.

Now, assume $n \geq 6$, and let $u \in J_n$ a monomial such that $u \notin I_n$. It follows that $u = x_1 x_n v$, with $v \in K[x_1, x_3, \dots, x_{n-2}, x_n]$. We can write $v = x_1^\alpha x_n^\beta w$, with $w \in K[x_3, \dots, x_{n-2}]$.

Since $u \notin I_n$, it follows that $w \notin (x_3x_4, \dots, x_{n-3}x_{n-2})$. Therefore, we have the S -module isomorphism:

$$\frac{J_n}{I_n} = x_1x_n \left(\frac{K[x_3, \dots, x_{n-2}]}{(x_3x_4, \dots, x_{n-3}x_{n-2})} \right) [x_1, x_n]$$

and therefore, by Lemma 1.2, Lemma 1.6 and [10, Lemma 3.6], we get $\text{sdepth}(J_n/I_n) = \text{depth}(J_n/I_n) = \lceil \frac{n-4}{3} \rceil + 2 = \lceil \frac{n+2}{3} \rceil$. \square

Remark 1.11. If $n = 4$, one can easily see that $\text{sdepth}(S/J_4) = 1$. Also, for $n = 7$, we can show that $\text{sdepth}(S/J_7) = 2$, see Example 2.5. On the other hand, using the *SdepthLib.coc* of *CoCoA*, see [14], we get $\text{sdepth}(S/J_{10}) = 4$ and $\text{sdepth}(S/J_{13}) = 5$. This remark, yields the following conjecture.

Conjecture 1.12. $\text{sdepth}(S/J_n) = \lceil \frac{n}{3} \rceil$, for all $n \geq 10$ with $n \equiv 1 \pmod{3}$.

Even if J_n and I_n are closely related, the difficulty of Conjecture 1.12 should not be underestimate. See for instance [2], where the authors, using fine tools of combinatorics were hardly able to compute the Stanley depth of the maximal monomial ideal (x_1, \dots, x_n) . In the second section we will give a possible approach to this problem, see Example 2.5.

2 Bounds for Sdepth of quotient of monomial ideals

Lemma 2.1. Let $n \geq 1$ and $0 \leq k \leq n$ be two integers and let $\mathcal{P} = \{\sigma \in 2^{[n]} \mid |\sigma| \leq k\}$. Then, there exists a partition $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [C_i, D_i]$ with $|D_i| = k$.

Proof. If $k = n$ or $k = 0$ there is nothing to prove. Assume $1 \leq k \leq n - 1$. Note that \mathcal{P} is the partition associated to $S/I_{n,k+1}$, where $I_{n,k+1}$ is the ideal generated by all the square free monomials of degree $k+1$. According to [7, Theorem 1.1], $\text{sdepth}(S/I_{n,k+1}) = k$. Thus, we can find a partition of \mathcal{P} , as required. \square

Proposition 2.2. Let $\mathcal{P} \subset 2^{[n]}$ be a poset such that $\text{sdepth}(\mathcal{P}) \geq k$. Then there exists a partition of \mathcal{P} , such that, for each interval $[C, D]$ of it, if $|C| < k$ then $|D| = k$.

In particular, the above assertion holds, if $I \subset J$ are two monomial square-free ideals such that $\text{sdepth}(J/I) = k$ and $\mathcal{P} = \mathcal{P}_{J/I} := \mathcal{P}_{S/I} \cap \mathcal{P}_J$.

Proof. According to Herzog, Vladoiu and Zheng [10], we have $\text{sdepth}(J/I) = \text{sdepth}(\mathcal{P}_{J/I})$. Since $\text{sdepth}(\mathcal{P}) \geq k$, we can find a partition of \mathcal{P} , such that each interval $[C, D]$ in this partition has $|D| \geq k$.

Let $[C, D]$ be an interval of the partition of \mathcal{P} . If $|C| \geq k$ or $|D| = k$ there is nothing to do. Assume $|C| < k$ and $|D| > k$. We denote $|C| = t$ and $|D| = s$. Without losing the generality, we may assume that $D = [s]$ and $C = [s] \setminus [s-t]$. Using the previous Lemma, we can find a partition of $[\emptyset, [s-t]] = \bigcup_{i=1}^r [\overline{C}_i, \overline{D}_i]$ with $|\overline{D}_i| = k-t$ whenever $|\overline{C}_i| < k-t$. Let $C_i = C \cup \overline{C}_i$ and $D_i = C \cup \overline{D}_i$. It follows that $[C, D] = \bigcup_{i=1}^r [C_i, D_i]$ is a partition with $|D_i| = k$, whenever $|C_i| < k$. If we apply this method for each interval in the partition of \mathcal{P} , finally, we will get a partition of \mathcal{P} , as required. \square

Corollary 2.3. *Let $\mathcal{P} \subset 2^{[n]}$ be a poset such that $\text{sdepth}(\mathcal{P}) \geq k$. Denote $\mathcal{P}_{\leq k} = \{\sigma \in \mathcal{P} \mid |\sigma| \leq k\}$. Then $\text{sdepth}(\mathcal{P}_{\leq k}) = k$.*

Proof. Obviously, $\text{sdepth}(\mathcal{P}_{\leq k}) \leq k$. According to Proposition 2.2, we can find a partition $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [F_i, G_i]$ of \mathcal{P} such that $|G_i| = k$, whenever $|F_i| < k$. Note that

$$[F_i, G_i] \cap \mathcal{P}_{\leq k} = \begin{cases} [F_i, G_i], & |F_i| < k, \\ [F_i, F_i], & |F_i| = k, \\ \emptyset, & |F_i| > k \end{cases}$$

Therefore, $\mathcal{P}_{\leq k} = \bigcup_{i=1}^r [F_i, G_i] \cap \mathcal{P}_{\leq k}$ is a partition of $\mathcal{P}_{\leq k}$ with its Stanley depth $\geq k$. \square

Let $\mathcal{P} \subset 2^{[n]}$ be a poset such that $\text{sdepth}(\mathcal{P}) \geq k$. We denote $\beta_t = |\{\sigma \in \mathcal{P} : |\sigma| = t\}|$, for all $0 \leq t \leq k$.

We consider the poset $\mathcal{P}_{\leq k} := \{\sigma \in \mathcal{P} : |\sigma| \leq k\}$. By Corollary 2.3, we can find a partition $\mathbf{P} : \mathcal{P}_{\leq k} = \bigcup_{i=1}^r [F_i, G_i]$ with $|G_i| = k$ for all i . We may assume that $|F_i| \leq |F_{i+1}|$ for all $i \leq r-1$. For all $0 \leq j \leq k$, we denote $\alpha_j = |\{i : |F_i| = j\}|$. Let $[F, G]$ be an arbitrary interval in the partition \mathbf{P} such that $|F| = j$ for some $j \leq k$. Note that in the interval $[F, G]$ we have exactly $\binom{k-j}{t-j}$ sets of cardinality t . Therefore, we get $\beta_t = \sum_{j=0}^t \binom{k-j}{t-j} \alpha_j$, for all $0 \leq t \leq k$. Moreover, $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1 - k\beta_0$, $\alpha_2 = \beta_2 - \binom{k}{2}\alpha_0 - (k-1)\alpha_1$ and so on. Thus, we proved the following Theorem.

Theorem 2.4. *If $\text{sdepth}(\mathcal{P}) \geq k$, then $\alpha_t \geq 0$ for all $0 \leq t \leq k$, where $\alpha_0 = \beta_0$ and $\alpha_t = \beta_t - \sum_{j=0}^{t-1} \binom{k-j}{t-j} \alpha_j$.*

Note that the above theorem give an upper bound for $\text{sdepth}(J/I)$, where $I \subset J$ are square free monomial ideals. Indeed, we can consider the poset $\mathcal{P} := \mathcal{P}_{J/I}$.

Example 2.5. We consider the poset $\mathcal{P} := \mathcal{P}_{S/J_n}$, where $J_n = (x_1x_2, \dots, x_{n-1}x_n, x_nx_1) \subset S$. We claim that $\beta_t = \binom{n-t+1}{t} - \binom{n-t-1}{t-2}$, for all $0 \leq t \leq n$.

Indeed, if $\sigma = \{i_1, \dots, i_t\} \in \mathcal{P}$ is a set of cardinality t such that $1 \leq i_1 < i_2 < \dots < i_t \leq n$, then $i_{j+1} \geq i_j + 2$ and $\{i_1, i_k\} \neq \{1, n\}$. There are exactly $\binom{n-t+1}{t}$, t -tuples $1 \leq i_1 < i_2 < \dots < i_t \leq n$ with $i_{j+1} \geq i_j + 2$ and exactly $\binom{n-t-1}{t-2}$, t -tuples $1 = i_1 < i_2 < \dots < i_t = n$ with $i_{j+1} \geq i_j + 2$. (To be more clear, if we denote $l_j := i_j - j + 1$, we have $1 \leq l_1 \leq l_2 \leq \dots \leq l_t \leq n - t + 1$ with $l_{j+1} > l_j$, and there are exactly $\binom{n-t+1}{t}$, t -tuples like this. If we fix $l_1 = 1$ and $l_t = n - t + 1$, we have $2 \leq l_2 \leq \dots \leq l_{t-1} \leq n - t$ and there are exactly $\binom{n-t-1}{t-2}$, $t-2$ -tuples like this).

Now, for $n = 7$, one can easily check that $\beta_0 = 1$, $\beta_1 = 7$, $\beta_2 = 14$ and $\beta_3 = 7$. For $k = 3$, we have $\alpha_0 = 1$, $\alpha_1 = 4$, $\alpha_2 = 2$ and $\alpha_3 = -1$. This shows, in the light of Theorem 2.4, that we cannot find a decomposition of the poset associated to S/J_7 with its Stanley depth equal to 3. On the other hand, by Proposition 1.8, we have $\text{sdepth}(S/J_7) \geq 2$, and thus $\text{sdepth}(S/J_7) = 2$.

For $n = 3k - 2$, where $k \geq 4$, we expect that $\alpha_0, \dots, \alpha_k$ are nonnegative, which is indeed the case for small values of k , using computer experimentation. However, this is useful only as an heuristic method to estimate the Stanley depth of S/J_n . In order to compute exactly this invariant, one has to produce a concrete partition of the associated poset.

In the second part of this section, we give a lower bound for the Stanley depth of a quotient of monomial ideals in terms of the minimal number of monomial generators. First, we recall several results.

Proposition 2.6. [4, Proposition 1.2] *Let $I \subset S$ be a monomial ideal (minimally) generated by m monomials. Then $\text{sdepth}(S/I) \geq n - m$.*

Proposition 2.7. [5, Remark 2.3] *Let $I, J \subset S$ be two monomial ideals. Then $\text{sdepth}((I + J)/I) \geq \text{sdepth}(J) + \text{sdepth}(S/I) - n$.*

Lemma 2.8. *Let $I, L \subset S$ be two monomial ideals such that L is minimally generated by some monomials w_1, \dots, w_s which are not in I . Then $\mathcal{B} = \{w_1 + I, \dots, w_s + I\}$ is a system of generators of J/I , where $J := L + I$.*

Proof. Denoting $G(I) = \{v_1, \dots, v_p\}$, it follows that $J = (v_1, \dots, v_p, w_1, \dots, w_r)$. So, if $w \in J \setminus I$ is a monomial, then $w_j | w$ for some $j \in [r]$ and therefore \mathcal{B} is a system of generators for J/I . On the other hand, since w_1, \dots, w_r minimally generate L , we get the minimality of \mathcal{B} . \square

We consider $I \subset J \subset S$ two monomial ideals. Denote $G(I) = \{v_1, \dots, v_p\}$ and $G(J) = \{u_1, \dots, u_q\}$ the sets of minimal monomial generators of I and J .

If $u_1 \in I$, then we may assume that $v_1 | u_1$. On the other hand, $I \subset J$ and therefore, there exists an index i such that $u_i | v_1$. We get $u_i | u_1$ and thus $u_i = u_1 = v_1$. Using the same argument, we can assume that there exists an integer $r \geq 0$ such that $u_1 = v_1, \dots, u_r = v_r$ and $u_{r+1}, \dots, u_q \notin I$. By Lemma 2.8, $\{u_{r+1} + I, \dots, u_q + I\}$ is a set of generators of J/I . With these notations, we have the following result, which is similar to [6, Theorem 2.4].

Proposition 2.9. $\text{sdepth}(J/I) \geq n - p - \lfloor \frac{q-r}{2} \rfloor$.

Proof. Denote $J' = (u_{r+1}, \dots, u_q)$. By our assumptions, we have $J/I = (I + J')/I$. By Proposition 2.7, it follows that $\text{sdepth}(J/I) \geq \text{sdepth}(J') + \text{sdepth}(S/I) - n$. By Theorem 1.4 and Proposition 2.6 we are done. \square

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